# Introduction to the Duality for Linear Programming

Let $(P)$ be

$$(P)\begin{cases}\text{maximize} & 2x_1 + 3x_2 \\ \text{s.t.} & 4x_1 + 8x_2 \le 12 \\ & 2x_1 + x_2 \le 3 \\ & 3x_1 + 2x_2 \le 4 \\ & x_1 \ge 0 \\ & x_2 \ge 0\end{cases}$$

**1:** Without solving $(P)$ itself, is it possible to provide an upper bound on the value of $(P)$ by using equation $4x_1 + 8x_2 \le 12$?

**Solution:** Yes - easily:

$$2x_1 + 3x_4 \le 4x_1 + 8x_2 \le 12$$

so the maximum is at most 12. We can even improve it by

$$2x_1 + 3x_4 \le \frac{1}{2}(4x_1 + 8x_2) \le 6.$$

This gives a maximum of at most 6.

**2:** Without solving $(P)$, is it possible to provide an upper bound on the value of $(P)$ using equations $4x_1+8x_2 \le 12$ and $2x_1 + x_2 \le 3$? *Hint: sum them*

**Solution:**

$$\begin{array}{rr} & 4x_1 + 8x_2 \le 12 \\ & 2x_1 + x_2 \le 3 \\ + & \overline{\phantom{6x_1 + 9x_2 \le 15}} \\ & 6x_1 + 9x_2 \le 15 \end{array}$$

Now we get $2x_1 + 3x_4 \le \frac{1}{3}(6x_1 + 9x_2) \le 5$.

**3:** Without solving $(P)$, how would you try to find the combination of constraints that provides the best upper bound? (solution might be another linear program, call it $(D)$)

**Solution:** We try to combine the three constraints (not the non-negativity constraints) and obtain an upper bound. Say the first constraints is multiplied by $y_1$, the second by $y_2$ and third by $y_3$.

So we have a combination of

$$\begin{aligned} &y_1 \cdot (4x_1 + 8x_2 \le 12) \\ &y_2 \cdot (2x_1 + x_2 \le 3) \\ &y_3 \cdot (3x_1 + 2x_2 \le 4) \end{aligned}$$

**What else does $y_i$ satisfy?** If $y_i < 0$, the inequality is reversed, so $y_i \geq 0$ We need the left hand sides to be at least $2x_1 + 3x_4$, hence

$$y_1 \cdot 4x_1 + y_2 \cdot 2x_1 + y_3 \cdot 3x_1 \geq 2x_1$$
$$y_1 \cdot 8x_2 + y_2 \cdot x_2 + y_3 \cdot 2x_2 \geq 3x_2$$

Next, we want to minimize the right hand side, which is $12y_1 + 3y_2 + 4y_3$. It gives a linear program $(D)$:

$$(D)\begin{cases} \text{minimize} & 12y_1 + 3y_2 + 4y_3 \\ \text{s.t.} & 4y_1 + 2y_2 + 3y_3 \geq 2 \\ & 8y_1 + y_2 + 2y_3 \geq 3 \\ & y_1, y_2, y_2 \geq 0 \end{cases}$$

- $(D)$ gives an upper bound on $(P)$
- $(P)$ gives a lower bound on $(D)$

**4:** Are solutions $\mathbf{x} = (\frac{1}{2}, \frac{5}{4})$ of $(P)$ and $\mathbf{y} = (\frac{5}{16}, 0, \frac{1}{4})$ for $(D)$ optimal solutions?

**Solution:** Yes! They are optimal solutions because they satisfy all constraints and values of the objective functions are the same.

**5:** Find the dual program $(D)$ to

$$(P)\begin{cases} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{cases}$$

**Solution:**

$$(D)\begin{cases} \text{minimize} & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & A^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{cases}$$

**6:** Find the dual program $(D)$ to

$$(P)\begin{cases} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{cases}$$

**Solution:** We first rewrite $A\mathbf{x} = \mathbf{b}$ as $A\mathbf{x} \geq \mathbf{b}$ and $-A\mathbf{x} \geq -\mathbf{b}$
Then we get

$$(P)\begin{cases} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq \mathbf{b} \\ & -A\mathbf{x} \geq -\mathbf{b} \\ & \mathbf{x} \geq 0 \end{cases} \qquad (D)\begin{cases} \text{maximize} & \mathbf{b}, -\mathbf{b}^T(\mathbf{u}, \mathbf{v}) \\ \text{s.t.} & A^T\mathbf{u} - A^T\mathbf{v} \leq \mathbf{c} \\ & \mathbf{u}, \mathbf{v} \geq 0 \end{cases}$$

Let $\mathbf{y} = \mathbf{u} - \mathbf{v}$. Then we can write

$$(D)\ \begin{cases} \text{maximize} & \mathbf{b},^T(\mathbf{y}) \\ \text{s.t.} & A^T\mathbf{y} \leq \mathbf{c} \end{cases}$$

Note that $\mathbf{y}$ can be negative.